Markov Methods

We define a Markov process by:

$$pr(B \mid X(\tau)=j \text{ and } A) = pr(B \mid X(\tau)=j)$$

where B is a future event ; A is past event $X(\tau)$ is the present state of the system.

or.

$$pr(A, X(\tau)=j, B)\, pr(X(\tau)=j) =$$
$$pr(A, X(\tau)=j)\, pr(X(\tau)=i, B)$$

There is complete symmetry between past and future. The Markov process is time reversible.

$$pr(X(\tau+u)=i_u \quad u=1\cdots s \mid X(\tau)=j, A)$$
$$= \prod_{0}^{s-1} pr(X(\tau+u+1)=i_{u+1} \mid X(\tau+u)=i_u)$$
$$= pr(X(\tau+u)=i_u \quad u=1\cdots s \mid X(\tau)=j).$$

dfn.

$$p_{ij} = pr(X(1)=j \mid X(0)=i)$$
$$= pr(X(\tau+1)=j \mid X(\tau)=i)$$

$$p_i = pr(X(0)=i)$$

$$p_i \geq 0 \qquad \sum p_i = 1$$
$$p_{ij} \geq 0 \qquad \sum_j p_{ij} = 1$$

We associate the matrix P with (p_{ij}) and the vector $p.$ with the p_i.

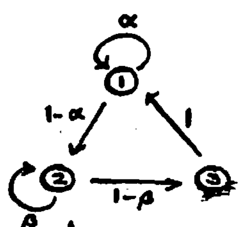
The component $(pP^n)_j = \mathrm{pr}(X(n) = j)$.

Inter communication

$i \sim j$ when $i = j$ or $i \neq j$ and $\exists$ M,N s.t.
$p_{ij}^m > 0$ $\quad p_{ji}^n > 0$

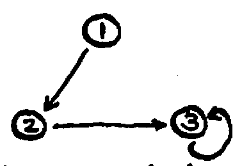
This is an equivalence relation.
note $\quad p_{ih}^{m+n} \geq p_{ij}^m \, p_{jh}^n$.

ex. $\begin{pmatrix} \alpha & 1-\alpha & 0 \\ 0 & \beta & 1-\beta \\ 1 & 0 & 0 \end{pmatrix}$



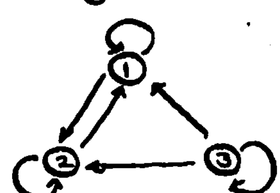
three classes, no one absorbing

$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.



three classes, ③ an absorbing state.

$\begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$



① and ② form a closed class

Persistance

We define the first passage probabilities:
$f_{ij}^n = \mathrm{pr}\{$ start at i reach j for the first time at $t = n \}$
$F_{ij} = \sum_{r \geq 1} f_{ij}^r = \mathrm{pr}\{$ start at i and reach j eventually $\}$

If $F_{ii} = 1$ we say class i is persistant.

pr{ state i is hit twice in $1 \le t \le n$, once being at $t = n$ } $= \sum_{1 \le r \le n} f_{ii}^r f_{ii}^{n-r}$

pr{ i hit at least s times } $= F_{ii}^s$

pr{ i hit exactly s times } $= F_{ii}^s - F_{ii}^{s+1}$

$= F_{ii}^s (1 - F_{ii})$

If a class is persistant then

pr{ i is hit finitely often } $= 0$

$F_{ii} < 1$ is the transient case

pr{ i is hit at least s times } $= F_{ii}^s \to 0$

as $s \to \infty$ so pr{ i hit ∞ often } $= 0$

$E(\text{no. of hits}) = \sum_1^\infty s F_{ii}^s (1 - F_{ii}) = F_{ii}/(1 - F_{ii})$

Dfn: $Y(n) = 1$ if $x(n) = i$

$= 0$ if otherwise

$\therefore E_i\left(\sum_{n \ge 1} Y(n)\right) = E_i(\text{no. of hits})$

$= \sum_{n \ge 1} E_i(Y(n)) = \sum_{n \ge 1} p_{ii}^n$

Hence i is persistant $\Leftrightarrow \sum_{n \ge 1} p_{ii}^n$ diverges

Thm 1 : Persistance and Transcience are class properties.

pf.

Suppose $i \sim j$ $p_{ij}^m > 0$ $p_{ji}^n > 0$

$\sum_r p_{ii}^{m+n+r} \geqslant \sum_r p_{ij}^m p_{jj}^r p_{ji}^n$

$\sum_r p_{jj}^{m+n+r} \geqslant \sum_r p_{ji}^n p_{ii}^r p_{ij}^m$

$\Rightarrow$ If $\sum p_{ii}^r$ converges so does $\sum p_{jj}^r$

Th^m 2:

Persistant classes are closed

pf:

Say $i \neq h$ and $p_{ih}^{m_0} > 0$ for some m_0; i persistant

$1 = pr_i\{ i \text{ hit } \infty \text{ often}\}$

$= \sum_h pr_i\{ x(m) = h \text{ and } i \text{ hit } \infty \text{ often}\}$

$= \sum_h pr_i\{ x(m) = h\} \, pr_h(\infty \text{ hits on } i)$

$=$

$$\sum_h p_{ih}^m = \sum_h p_{ih}^m \, pr_h(\infty \text{ hits on } i)$$

$$\sum_h p_{ih}^m (1 - pr_h(\infty \text{ hits on } i)) = 0 \quad \forall m$$

in particular for m_0. But $p_{ih}^{m_0} > 0$

$\therefore$

i persistant $\Rightarrow$ $pr_h(\infty \text{ hits on } i) = 1$

i persistant $i \sim h \Rightarrow h \sim i$

$\therefore$ $i \sim h$ ie i and h are in the same class.

Davidson's Inequality

$$1 - p_{ii}^{m+n} \geq p_{ii}^m (1 - p_{ii}^n)$$

pf:

$$\begin{aligned}
1 - p_{ii}^{m+n} &= pr(\text{start at } i, \text{ not at } i \text{ at } m+n) \\
&= \sum_\alpha pr_i(X(m) = \alpha \text{ and not } i \text{ at } m+n) \\
&= \sum_\alpha pr_i(X(m) = \alpha) \, \underset{\alpha \text{ at } m}{pr_\alpha}(\text{not } i \text{ at } m+n) \\
&\geq \text{term with } \alpha = i \\
&= p_{ii}^m (1 - p_{ii}^n)
\end{aligned}$$

$$\begin{aligned}
\therefore \quad 1 - p_{ii}^{m+n} &\geq p_{ii}^m (1 - p_{ii}^n) \\
&\geq p_{ii}^n (1 - p_{ii}^m)
\end{aligned}$$

Assume $p_{ii}^m \geq p_{ii}^n$ or $p_{ii}^m < p_{ii}^n$

$p_{ii}^m(1 - p_{ii}^n) \geq p_{ii}^m(1 - p_{ii}^m)$ $\quad p_{ii}^m \geq p_{ii}^n$

$p_{ii}^n(1 - p_{ii}^m) \geq p_{ii}^m(1 - p_{ii}^m)$ $\quad p_{ii}^m < p_{ii}^n$

Hence $1 - p_{ii}^{m+n} \geq p_{ii}^m(1 - p_{ii}^m)$.

Suppose i is persistant

$\sum_\alpha p_{i\alpha}^m (1 - pr_\alpha(\infty \text{ many hits to } i))$

$= 1 - \sum_\alpha pr_i(X(m) = \alpha \text{ and } \infty \text{ many hits to } i)$

$= 1 - pr_i(\infty \text{ many hits to } i) = 1 - 1 = 0$

So $\forall\, m, \alpha$ $\quad p_{i\alpha}^m(1 - pr_\alpha(\infty \text{ many hits to } i)) = 0$

Suppose i persistant and $i \sim j$

put $\alpha = j$ m s.t. $p_{ij}^m \neq 0$

$\therefore$ $pr_j(\infty \text{ many hits to } i) = 1 \Rightarrow j \sim i$

So a persistant class is closed.

Within a persistant class, if $i, j \in \mathcal{A}$

then $pr_i(\infty \text{ many hits to } j) = 1$.

1, j. Suppose state space finite

A_{ij} = 'start at i, ∞ hits on j'

$\bigcup_j A_{ij}$ = 'start at i ∞ hits somewhere'

$P(\bigcup_j A_{ij}) = 1 \leq \sum_j pr_j(A_{ij})$

So given i given finite state space.

$\exists j$ s.t. $pr_i(A_{ij}) > 0$

$pr_i(\infty \text{ hits on } j) > 0$

$$\sum_{m\geq 1} pr_i(j \text{ avoided for } 1\leq t\leq m,\ X(m)=j,\ \infty \text{ hits on } j)$$

$=$

$$\sum_{m\geq 1} pr_i(j \text{ avoided for } 1\leq t\leq m,\ X(m)=j) =$$

$$F_{ij}\, pr_j(\infty \text{ hits on } j) > 0$$

given i and state space finite.

$\Rightarrow \exists j$ s.t. j is persistant ; $i \rightsquigarrow j$

Hence when the S/S is finite then closed class $\equiv$ persistant class.

This is false if the S,S is infinate.

$$\begin{array}{l} \left[\begin{array}{ccccc} p_1 & q_1 & & & \\ p_2 & 0 & q_2 & & 0 \\ p_3 & 0 & 0 & q_3 & \\ \vdots & & 0 & & \ddots \end{array}\right. \end{array} \qquad \sum p_j < \infty$$

Here the whole matrix is a closed class.
But it is not persistant.

$q_1 \cdots q_n \cdots = \prod_j^{\infty}(1-p_j) > 0$ so we do not necessarily return to state 1.

Let C be a closed class.

$\alpha^C_i = pr_i(\text{trapped in } C)$

α^C is the vector of absorption probabilities.

$\alpha_i^c \geq 0 \quad \forall\, i$

$P\alpha_i^c = \alpha_i^c$ since

$$\sum_h p_{ih}(\alpha^c)_h = \sum_h pr_i(X(1)=h)\, pr_h(\text{trapped in } C)$$
$$= \sum_h pr_i(X(1)=h \text{ and trapped in } C)$$

$$= \sum \quad = pr_i(\text{trapped in } C) = \alpha_i^c .$$

$\therefore \quad y_i = 1$ on C

$y_i \geq 0 \quad \forall\, i$

$Py = y$

Th^m : α^c is the minimal solution of the above

pf:

$$y_i = \sum_{\substack{\alpha \\ \alpha \in \text{ non-closed classes}}} p_{i\alpha}^n y_\alpha + \sum_{\substack{\rho \\ \rho \in \text{ closed classes}}} p_{i\rho}^n y_\rho + \sum_{\substack{j \\ j \in C}} p_{ij}^n y_j$$

$$\therefore \; y_i \geq \sum_j p_{ij}^n y_j = \sum_j p_{ij}^n$$

$pr_i(X(n) \in C) \leq y_i$

$pr_i(\text{trapped in } C) \leq y_i$

$\alpha_i^c \leq y_i$

$$\left[\begin{array}{ccccc}
1 & 0 & 0 & 0 & \\
p_1 & 0 & q_1 & & O \\
p_2 & 0 & 0 & q_2 & \\
p_3 & 0 & 0 & 0 & q_3 \\
\vdots & & O & & \ddots
\end{array}\right]$$

like climbing a ladder.

$y_0 = 1$

$y_i \geqslant 0$

$y_i = p_i + q_i y_{i+1} \qquad \forall \ i > 1$

$q_i (y_{i+1} - 1) = y_i - 1$

$$y_i - 1 = \frac{1}{q_{i-1} \cdots q_1} (y_1 - 1)$$

$$y_i = 1 + \frac{y_1 - 1}{\prod_1^{i-1} q_n}$$

$$y_i = 1 + \frac{y_1 - 1 + \Pi_1^{\infty}}{\Pi_1^{i-1}} - \frac{\Pi_1^{\infty}}{\Pi_1^{i-1}}$$

$$\frac{y_1 - 1 + \Pi_1^{\infty}}{\Pi_1^{i-1}} \geqslant 0$$

It can be zero as we can take

$y_1 = 1 - \prod_{j \geq 1} (1 - p_j)$

So this y_1 must give the minimum

$\text{pr}_i(\text{absorption}) = 1 - (q_1 q_2 \cdots)$.

$$\left| \begin{array}{ccccc} p_0 & q_0 & & & \\ p_1 & 0 & q_1 & & \\ p_2 & 0 & & q_2 & \\ \vdots & & & & \ddots \end{array} \right. \qquad \Sigma p_j < \infty$$

Here we have one large class.

Is it persistent ?

We freeze at state zero. If state zero is persistant. Then

pr (absorption) in

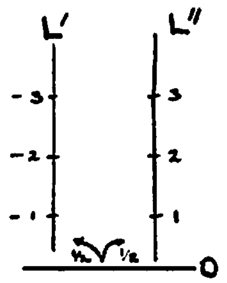
$$
= \quad
\begin{array}{|cccc}
\hline
1 & 0 & & \\
p_1 & 0 & q_1 & \\
p_2 & 0 & 0 & q_2 \cdots
\end{array}
$$

pr (return to p_0) $\longrightarrow$

$$
\begin{array}{|cccc}
\hline
p_0 & q_0 & & \\
p_1 & 0 & q_1 & \\
p_2 & 0 & 0 & q_2
\end{array}
$$

We have seen that the first is < 1

$\therefore$ pr (return to p_0) $< 1 \Rightarrow$ the class is not persistant.



has matrix:

$$
\begin{array}{c|c|c}
 & \overset{0}{\downarrow} & \\
O & P_2 & 0 \quad 0 \quad q_2 \cdots \\
 & P_1 & 0 \quad q_1 \quad 0 \\
\hline
0 \rightarrow \quad \frac{1}{2} & 0 & \frac{1}{2} \\
\hline
q_1 \quad 0 & P_1 & \\
\cdots q_2 \quad 0 \quad 0 & P_2 & O
\end{array}
$$

A: eventually $\Uparrow$ in L'

B: eventually $\Uparrow$ in L'' ($\Uparrow$: goes up forever)

$pr_i(A) + pr_i(B) \leq 1$.

Consider the system

$$\begin{tikzcd}
0 \arrow[r, shift left, "\frac{1}{2}"'] \arrow[d, shift right, "\frac{1}{2}"'] & 1'' \arrow[l, shift left] \arrow[dr] & \\
1' \arrow[u, shift right] \arrow[dr] & & \infty'' \\
& \infty' &
\end{tikzcd}$$

$0, 1', 1''$ form a class, transient

∞', ∞'' are closed absorbing classes

	0	$1'$	∞'	$1''$	∞''
0		$\frac{1}{2}$		$\frac{1}{2}$	
$1'$	$1-\Pi$	$\cdot$	Π		
∞'			1		
$1''$	$1-\Pi$				Π
∞''					1

where $\Pi = \prod_{\alpha \geq 1} q_\alpha$

$y_i = P_{r_i}(\text{abs. at } \infty')$ $\qquad y > 0$

$y_0 = \frac{1}{2} y_{1'} + \frac{1}{2} y_{1''}$ $\qquad y_{\infty'} = 1$

$y_{1'} = (1-\Pi) y_0 + \Pi . 1$ $\qquad y_{\infty''} = 0$

$y_{1''} = (1-\Pi) y_0 + 0$ $\qquad Py = y$

$y_0 = \frac{1}{2}(2(1-\Pi) y_0 + \Pi)$

$y_0 = y_0(1-\Pi) + \frac{1}{2}\Pi$

$y_0 = \frac{1}{2} \quad \Rightarrow$

$P_0(\Uparrow L') + P_0(\Uparrow L'') = 1$

Hence the walk is certain to stay on one ladder after some point.

Strong Markov Theorem

pr_i (returns to i and immediately after the first such return and goes j)

$= \sum_{n \geq 1} pr_i (\neq i \text{ up to } n, X(n) = i, X(n+1) = j)$

$= \sum_{n \geq 1} pr_i (\neq i, X(n) = i) p_{ij}$

$= F_{ii} p_{ij}$

Now consider:

pr_i (hits i but only finitely often and after the last hit goes to k immediately)

$= \sum_{n \geq 1} pr_i (X(n) = i, X(n) = k, \neq i)$

$= \sum_{n \geq 1} pr_i (X(n) = i) pr_i (X(1) = k, \neq i)$

$\neq pr_i (\text{hits } i \text{ finitely often}) pr_i (X(1) = k, \neq i)$

Dfn: Markov time.

This is a r.v. taking values $0, 1, 2, \cdots, \infty$ s.t.

$\forall n$ the statement "$T = n$" is a statement about $X(0), X(1), \ldots, X(n)$

Theorem: Let T be a Markov time; let A be a pre-T event. Let B be a post T event.

$\theta(i_1, i_2, \ldots) = (i_2, i_3, \ldots)$

So there is some set of paths

E s.t. $B = \theta^{-T} E$

"$T=n$ and B" = "$T=n$ and $\theta^{-T}E$".

$$pr(B \mid A \text{ and } X(T) = j) = pr_j(E)$$

$T=n$ is a statement about $X(0), \ldots, X(n)$

A and $T=n$ " "

B and $T=n$ = $T=n$ and $\theta^{-1}E$

We say A, B are pre and post T events.

$pr(A, X(T)=j, B) =$

$\sum_{n \geq 0} pr(A, T=n, X(n)=j, T=n, B) =$

$\sum_{n \geq 0} pr(A, T=n, X(n)=j)\, pr_j(E)$ where $E = \theta B$ $=$

$pr(A, T<\infty, X(T)=j)\, pr_j(E) =$

$pr(A, X(T)=j)\, pr_j(E)$.

ex.

pr_i(exactly m hits on state j). Prove

$= F_{ij} F_{jj}^{m+1}(1-F_{jj})$

Df^n. $\hat{z}_{ij} = E_i$(no. of hits on j not preceded by a hit on i)

$\hat{z}_{ii} = 1$ if i persistant

$= 0$

$\therefore \hat{z}_{ii} = F_{ii} \leq 1$

$$\hat{z}_{ij} = E_i\left(\sum_{n\geq 1} J(n)\right)$$

where $J(n) = 1$ if $X(n) = j$ with no previous hits on i
$= 0$ otherwise

$$= \sum_{n\geq 1} E_i(J(n)) = \sum_{n\geq 1} pr_i(\neq i, X(n) = j)$$

$$= p_{ij} + \sum_{\alpha\neq i} p_{j\alpha} p_{\alpha j} + \sum_{\alpha\neq i}\sum_{\beta\neq i} p_{i\alpha} p_{\alpha\beta} p_{\beta j} + \cdots \leq \infty$$

If $i \neq j$ and are in the same class then
$0 < \hat{z}_{ij} < \infty$

$$\hat{z}_{ij} F_{ji}^{m} = \sum_{n\geq 1} pr_i(\neq i, X(n) = j) F_{ji}^{m}$$

$$= \sum_{n\geq 1} pr_i(\neq i, X(n) = j, \neq i, X(n+m) = i)$$

$$\leq \sum_{n\geq 1} pr_i(\neq i, X(n+m) = i) = \sum_{n\geq m} F_{ii}$$

$$= F_{ii} \leq 1$$

$x > 0$ $xF \leq x$ left hand inequality.
$x_i p_{ij} + \sum_{\alpha\neq i} x_\alpha p_{\alpha j} \leq x_j$
so
$x_i\left(p_{ij} + \sum_{\beta\neq i} p_{i\beta} p_{\beta j} + \cdots\right) \leq x_j$ by induction
$x_i \hat{z}_{ij} \leq x_j$
$x_a \hat{z}_{a\cdot} \leq x_a$

Now def^n $z_{ij} = \begin{cases} 1 & i = j \\ \hat{z}_{ij} & i \neq j \end{cases}$

$$(z_{a\cdot} P)_j = z_{aa} p_{aj} + \sum_{\alpha\neq a} \hat{z}_{a\alpha} p_{\alpha j}$$

$= p_{aj} + \sum_{\alpha \neq a} \hat{z}_{a\alpha} p_{\alpha j} = \hat{z}_{aj} \leq z_{aj}$

So $z_{a\cdot}$ is the minimal x s.t.

$x_a = 1 \quad x \geq 0 \quad xP \leq x$

$\forall a$

$(z_{a\cdot} P)_j = z_{aj} \quad j \neq a$

$(z_{a\cdot} P)_a < z_{aa}$ if a transient

$= z_{aa}$ if a persistent.

Suppose all states belong to one persistant class

$z_{a\cdot} P = z_{a\cdot}$

Ratio ergodic theorem

$i \longrightarrow j \longrightarrow j \longrightarrow j$

how many hits on k for each arc.

hits on j =

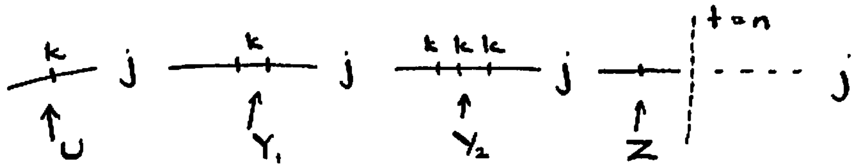
$H_n(k|i)$ = (hits on k starting at i up to n)

$H_n(j|i)$ = (" j " ")

as $n \longrightarrow \infty$

$$\frac{H_n(k|i)}{H_n(j|i)} \longrightarrow \frac{\mu_k}{\mu_j}$$

$$\frac{\text{time spent in } k}{\text{time spent in } j} \longrightarrow \frac{\mu_k}{\mu_j}$$

i) 

U, Y_1, Y_2 are no. hits on k.

Let $H_n(j \mid i) = w+1$

$$\frac{H_n(k \mid i)}{H_n(j \mid i)} = \frac{U + (Y_1 + \cdots + Y_w) + Z}{W+1}$$

We use the SLLN on $Y_1, Y_2, \cdots$ IID r.vs.

$E|Y_1| < \infty$

As $\underset{n\to\infty}{}$ $\dfrac{Y_1 + \cdots + Y_n}{n} \to E(Y)$ with probability 1.

$$\frac{Y_1 + \cdots + Y_W}{W} \cdot \frac{W}{W+1} \longrightarrow \frac{\mu_k}{\mu_j}$$

$E(Y_1) = z_{jk} = \mu_k / \mu_j < \infty$

$$\frac{U}{W+1} \to 0$$

$$\frac{Z}{W+1} \leq \frac{Y_{w+1}}{W+1} = \left(\frac{Y_1 + \cdots + Y_{w+1}}{w+1}\right) - \frac{W}{W+1}\left(\frac{Y_1 + \cdots + Y_w}{w}\right)$$

$\to 0$

$$\therefore \frac{H_n(k \mid i)}{H_n(j \mid i)} \longrightarrow \frac{\mu_k}{\mu_j} \quad \text{a.s.}$$

Consider

$p_{ij}^n = pr_i(X(n) = j)$

We use the Abel sum:

$$\sum_{n\geq 0} p_{ij}^n s^n (1-s) \qquad 0\leq s<1$$

$\lim_{s\uparrow 1} (1-s)\sum_{n\geq 0} p_{ij}^n s^n$ if it converges is called the Abel $\lim_{n\to\infty} p_{ij}^n$

$$p_{ij}^n = pr_i(X(\cdot) \text{ hits } j \text{ first at } t \quad 1\leq t\leq n \ ; X(n)=j)$$

$$= \sum_{t=1}^{n} pr_i(X \text{ hits } j \text{ first at } t \ ; \ X(n)=j)$$

$$= \sum_{t=1}^{n} pr_i(\neq j, X(t)=j, X(n)=j)$$

$$= f_{ij}\, p_{jj}^{n-1} + \cdots + f_{ij}^{n-1} p_{jj} + f_{ij}^n$$

$p_{ij} = f_{ij}$

$p_{ij}^{n+1} = (f_{ij}\, p_{jj}^n + \cdots + f_{ij}^n p_{jj}) + f_{ij}^{n+1}$

put $i=j$

$p_{jj} = f_{jj}$

$p_{jj}^{n+1} = (f_{jj}\, p_{jj}^n + \cdots + f_{jj}^n p_{jj}) + f_{jj}^{n+1}$

Let $P_{ij}(s) = \sum_{n\geq 1} p_{ij}^n s^n$

$$0\leq s<1$$

$F_{ij}(s) = \sum_{n\geq 1} f_{ij}^n s^n$

$F_{ij}(1)\leq 1 \qquad P_{ij}(1)\leq\infty$.

$P_{ii}(s) = F_{ii}(s) + F_{ii}(s)P_{ii}(s)$

$P_{ii}(s) = F_{ii}(s)/(1-F_{ii}(s))$

$F_{ii}(s) = P_{ii}(s)/(1+P_{ii}(s))$

$P_{ij}(s) = F_{ij}(s) + F_{ij}(s)P_{jj}(s)$

$P_{ij}(s) = F_{ij}(s)/(1 - F_{jj}(s))$

$\lim_{s \to 1} (1-s)P_{ij}$ is the limit we are trying to examine

$$\frac{1 - F_{jj}(s)}{1-s} = \frac{1 - F_{jj}(s) + F_{jj}(s) - F_{ij}(s)}{1-s}$$

$$= \frac{1 - F_{jj}(s)}{1-s} + \sum_{n \geq 1} f_{jj}^{n}(1 + s + s^2 + \cdots + s^{n-1})$$

what happens as $s \uparrow 1$.

$$\rightarrow \left\{ \begin{matrix} 0 & \text{pst} \\ \infty & \text{trt} \end{matrix} \right\} + \sum_{n \geq 1} n f_{jj}^{n} = \sum_{1}^{\infty} n f_{jj}^{n}$$

$= \infty f_{jj}^{\infty}$

$= E_j(T_j) = m_j$

$(1-s)P_{ij}(s) \longrightarrow F_{ij}(s)/m_j = \pi_{ij}$

So the abel limit alway exists

(i) say 2nd state, j, transient $\Rightarrow \pi_{ij} = 0 \ \forall i$

(ii) " " persistent

$\pi_{ij} = \alpha_j^{c(j)}/m_j$ where $\alpha_i^{c(j)}$ = prob. of being absorbed from i into the persistant class to which j belongs.

How do we compute m_j ?

Write $\lambda_j = 1/m_j$

We consider persistant j only, since j transient $\Rightarrow$ $\lambda_j = 0$

We consider one persistant class:

$\pi_{ij} = \pi_{jj}$ $\quad \Pi = (\pi_{ij})$ $\quad (\pi_{ij})$ has identical rows.

$\mathbf{1} = (1, 1, \dots 1)$

~~$P(s) =$~~

~~$\sum_{n \geq 1} p^n s^n \quad 0 \leq s < 1 \quad = \quad R(s)$~~

$Q(s) = \sum_{n \geq 1} p^n s^n \quad 0 \leq s < 1$

$R(s) = (1-s)\, Q(s)$

$(R(s))_{ij} \rightarrow \pi_{ij} \qquad R(s) \rightarrow \Pi$ as $s \rightarrow 1$.

$$s P R(s) = (1-s) s P Q(s) = s R(s) P$$
$$= \cancel{P s R(s)}$$

$$s P R(s) = R(s) s P$$
$$= R(s) - (1-s) s P$$

$s \uparrow 1$

$$P s R(s) = R(s) - (1-s) s P$$
$$= \Pi - 0$$

$P s R(s) \rightarrow P \Pi$

$\therefore \quad P \Pi = \Pi$

$s R(s) P = R(s) - (1-s) s P$ take limits

$\therefore \quad \Pi P \leq \Pi \qquad$ since $\sum_{\alpha} p_{\alpha j} \leq \infty$

Thus we have

$P\Pi = \Pi \qquad \Pi P \leq \Pi$

$\Pi = 1'\lambda$

$(P1')\lambda = 1'\lambda$ gives $1'\lambda = 1'\lambda$

$(1'\lambda P) \leq 1'\lambda$ gives

$1_i(\lambda P)_j \leq 1_i\lambda_j \quad : \quad \lambda P \leq \lambda$.

But the class was persistant so $\lambda P = \lambda$
and λ must be a multiple of μ.

each $\mu_j > 0 \quad \mu P = \mu$

(i) Suppose $\Sigma\mu_j < \infty$

$\mu P^n = \mu \qquad \mu\, Q(s) = \mu s/(1-s)$

$\qquad\qquad \mu \cdot R(s) = \mu s$

take limit : $\mu\Pi = \mu$

$\mu(1'\lambda) = \mu \qquad (\mu 1')\lambda = \mu$ giving

$\Sigma\mu_j\lambda_j = \mu_j$

$\lambda_j = \mu_j / \Sigma\mu_j$

corrollary $\Sigma\lambda_j = 1$

$\pi_{ij} = \pi_{jj} = \lambda_j = \frac{1}{m_j}$, so if all states form a persistant class.

$\sum_j \pi_{ij} = 1$. So Π is a stochastic matrix.

$\Sigma \frac{1}{m_j} = 1 \quad : \quad \Sigma \frac{1}{\text{mean recurrance time}} = 1$

(ii) Suppose $\Sigma\mu_j = \infty$

$\mu . R(s) = s\mu$ we get

$\mu\Pi \leq \mu \qquad \underset{\substack{\uparrow \\ \infty}}{\Sigma\mu_j\lambda_j} \leq \underset{\substack{\uparrow \\ \text{pos. finite}}}{\mu_j} \quad \Rightarrow \quad \lambda_j = 0$

Thus in a persistant class.

$\Sigma \mu_j < \infty \qquad \lambda_j = \mu_j / \Sigma \mu_j$ Positive

or

$\Sigma \mu_j = \infty \qquad \lambda_j = 0$ Null

In the positive case Π is a strictly positive stochastic matrix.

Continuous Time Markov Processes

We could define discrete time by
$P(\cdot) : \mathbb{Z} \xrightarrow[\text{hm}]{} \Sigma$ (semi-group of stocastic matrices)

$P(0) = I$
$P(m+n) = P(m)P(n) \qquad P_1(P_2 P_3) = (P_1 P_2)P_3$

The extention to continuous time is
$P(\cdot) : T_+ \xrightarrow[\text{HOM}]{} \Sigma$

$P(0) = 1$ axiom 1
$P(t)$ is a stochastic matrix axiom 2
$P(t+s) = P(t)P(s)$ axiom 3
$P(t) \to I = P(0)$ as $t \to 0$ axiom 4
ie $p_{ij}(t) \to \delta_{ij}$ as $t \to 0 \quad \forall i,j$
It is sufficient for this that
$p_{ii}(t) \to 1$ as $t \to 0 \quad \forall i$ since we are working with stochastic matrices.

Levy's Containing Lemma

$$p_{ij}(t+\delta) - p_{ij}(t) = \sum_\alpha p_{i\alpha}(\delta)p_{\alpha j}(t) - p_{ij}(t)$$

$$= \sum_{\alpha \neq i} + \sum_{\alpha = i} - p_{ij}(t)$$

$$\leq \sum_{\alpha \neq i} p_{i\alpha}(\delta) + p_{ii}(\delta)p_{ij}(t) - p_{ij}(t)$$
$$= 1 - p_{ii}(\delta) - p_{ij}(t)[1 - p_{ii}(\delta)]$$
$$\leq (1 - p_{ij}(t))(1 - p_{ii}(\delta))$$

also

$\geq p_{ii}(\delta)\, p_{ij}(t) - p_{ij}(t) = -p_{ij}(t)(1 - p_{ii}(\delta))$

$\therefore$

$| p_{ij}(t+\delta) - p_{ij}(t) | \leq (1 - p_{ii}(\delta))$

$| p_{ij}(s) - p_{ij}(t) | \leq 1 - p_{ii}(|s-t|)$
which is the Levy inequality.

It implies $p_{ij}(\cdot)$ is uniformly continuous on the whole of the real line.
It is also uniform w.r.t. the second state j.

Positivity Lemma

$p_{ii}(t) \geq p_{ii}(\tfrac{t}{2})\, p_{ii}(\tfrac{t}{2}) \geq \cdots \geq \{ p_{ii}(\tfrac{t}{2^k}) \}^{2^k}$
fix i and t
chose k large enough so $p_{ii}(\tfrac{t}{2^k}) > 1/17$ say
then $p_{ii}(t) > 0 \quad \forall i \ \forall t$

$p(n\delta) = p(\delta)^n$ a δ-skeleton.

Differentiability

$i = j \qquad p_{ii}(t) > 0$
$p_{ii}(t) = e^{-x(t)} \qquad 0 \leq x(t) < \infty$
$x(0) = \lim_{t \to 0} x(t) = 0$
$x(t+s) \leq x(t) + x(s)$

Let $0 < \tau < t < \infty$

$t = N\tau + s \qquad N \in \mathbb{Z}^+ \qquad 0 \leq s < \tau$

$$\frac{x(t)}{t} \leq \frac{Nx(\tau)}{t} + \frac{x(s)}{t}$$

$$\leq \frac{(t-s)}{t} \cdot \frac{x(\tau)}{\tau} + \frac{x(s)}{t}$$

let $\tau \to 0$

$x(s)/t \to 0$

$x(t)/t \leq \liminf\limits_{\tau \downarrow 0} x(\tau)/\tau \qquad *$

Let $t \to 0$

$$\limsup_{t \downarrow 0} \frac{x(t)}{t} \leq \lim_{\tau \downarrow 0} \inf \frac{x(\tau)}{\tau}$$

$\Rightarrow \lim\limits_{t \downarrow 0} \dfrac{x(t)}{t} = q$ exists

and from $*$ $\quad x(t) \leq qt \quad \forall t.$

$0 \leq e^{-\alpha} \leq 1$

$0 \leq 1 - e^{-\alpha} \leq \alpha$

$0 \leq \alpha - 1 + \alpha^{-\alpha} \leq \alpha^2/2$

$\alpha(1 - \frac{\alpha}{2}) \leq 1 - e^{-\alpha} \leq \alpha$

$$\frac{x(t)}{t}\left(1 - \frac{x(t)}{2}\right) \leq \frac{1 - p_{ii}(t)}{t} \leq \frac{x(t)}{t} \to q$$

$\therefore \dfrac{1 - p_{ii}(t)}{t} \to q_i < \infty$ as $t \to 0$

$p_{ii}(t) \geq e^{-q_i t}$

By Levy

$|p_{ij}(t) - p_{ij}(s)| \leq 1 - e^{-q_i|t-s|}$

We can have $q_i = \infty$, but in few practical examples.

$p_{ii}'(0)$ exists $= -q_i = q_{ii}$ anticipating $p_{ij}'(0) = q_{ij}$ which exists, is never infinate. (very hard to prove).

$\sum_{\substack{\alpha \neq i \\ \alpha \leq N}} p_{i\alpha}(t) \leq 1 - p_{ii}(t)$

$\sum_{\substack{\alpha \neq i \\ \alpha \leq N}} q_{i\alpha} \leq q_i \qquad \forall N$

$\sum_{\alpha \neq i} q_{i\alpha} \leq q_i$

$<$ can happen

$=$ " "

$\infty = \infty$ " "

finite $< \infty$ " "

Further conditions:

A) $q_i < \infty \quad \forall i$ "stable"

$p_{ij}(t) = \delta_{ij} + q_{ij} \cdot t + o(t)$

B) $\sum_{\alpha \neq i} q_{i\alpha} = q_i$ ie. Q has zero row sums

called "conservative"

$\sum_{\alpha}$ flow of probability from i to α =

flow of probability out of i.

Consider $i = 0, 1, 2, \ldots$

i	j	q_{ij}
0	1	$\kappa > 0$
$i \geq 1$	$i+1$	$\kappa + \lambda i$
	$i-1$	$R\ \mu i$

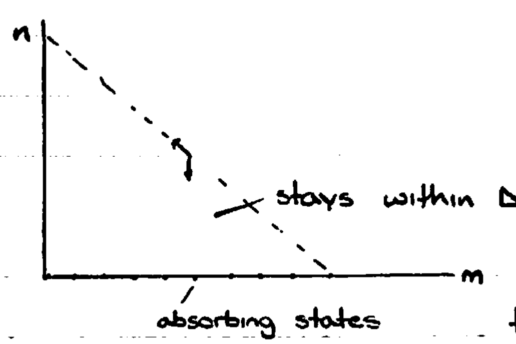
a birth-death process

$\lambda, \mu \geq 0$

If we confine discussion to A), B) then a system with this Q matrix exists.
C) makes it unique.

an epidemic process
$m =$ no. of susceptibles
$n =$ no. actively infected
$i = (m,n)$

i	j	q_{ij}
$(m,0)$	absorbing state	
$n \geq 1$ (m,n)	$n \to n-1$ $m \to m$	γn
$n \geq 1$ (m,n) $m \geq 1$	$m \to m+1$ $n \to n+1$	βmn



there are only finitely many states.

Finitely many states
Existence
Let $P(t) = \exp(Qt)$
We then get all the proper properties for $P(t)$
$P(u)P(v) = P(u+v)$ etc.
$P(t)$ is stochastic since:

$$P(t)\mathbf{1} = \left(I + tQ + \frac{t^2}{2} Q^2 + \dots \right)\mathbf{1}$$

$$= I\mathbf{1} = \mathbf{1} \quad \text{unity row sums.}$$

Consider $cI + Q$. Make c larger than any of the q_i so $cI + Q$ is non-negative.

$$P(t) = \exp\{-ctI + t(cI + Q)\}$$

$= e^{-ct} \exp\{ t(cI + Q) \} \geq 0$

$\therefore \; p_{ij}(t) \geq 0$

Finite states:

chose $\tau > 0$ s.t. $\tau q_i \leq 1 \quad \forall i$

$$P(t) = \exp\{ -t/\tau \, I + t/\tau \underbrace{(I + cQ)}^{R} \}$$

$$= \sum_{n \geq 0} e^{-t/\tau} \frac{(t/\tau)^n}{n!} R^n$$

much better than

$$P(t) = \sum_{n \geq 0} \frac{t^n}{n!} Q^n$$

c) $Qy = y$ has no bounded solution except $y = 0$.

Finitely many states.

$$\sum_{\alpha} q_{i\alpha} y_\alpha = y_i$$

$$\sum_{\alpha \neq i} q_{i\alpha} y_\alpha - q_i y_i = y_i$$

$$(1 + q_i) y_i = \sum_{\alpha \neq i} q_{i\alpha} y_\alpha$$

$\exists j$ s.t. $|y_j| = \max_{\forall i} |y_i|$

$$(1 + q_j)|y_j| \leq \sum_{\alpha \neq i} q_{i\alpha} |y_\alpha| \leq \sum_{\alpha \neq i} q_{i\alpha} |y_j|$$

If $y \neq 0 \quad y_j > 0$

$(1 + q_j) \leq q_j \quad \#$ $\quad \therefore \; y = 0$

$p_{ij}(n\delta) = [p_{ij}(\delta)]^n_{ij}$

δ- skeleton of the process

Q , q_{ij} . No matter how small we make δ can we make the continuous problem discrete.

(A,B) - Q matrix.

Residence times

pr_i (stays in i up to time t)

$= \lim_{n\to\infty} pr_i$ (in i at each of the times $\frac{mt}{2^n}$)

where $m = 0,1,2,\cdots,2^n$.

$$= \lim_{n\to\infty} \left\{ p_{ii}\left(\frac{t}{2^n}\right) \right\}^{2^n+1}$$

$$= \lim_{n\to\infty} \left\{ 1 - \frac{t}{2^n} \frac{(1-p_{ii}(t/2^n))}{t/2^n} \right\}^{2^n+1}$$

apply $(1-\frac{x_m}{m})^m \to e^{-x}$ if $x_m \to x$

$= e^{-q_i t}$

$p(T_i \le t) = 1 - e^{-q_i t}$

$p(T_i > t) = e^{-q_i t}$

Let Z be Q with zeros on the main diagonal

Let S be : if $q_i = 0$ $s_{ij} = \delta_{ij}$

$q_i > 0$ $s_{ii} = 0$

$s_{ij} = q_{ij}/q_i$

S is stochastic.

$\sum_{n\ge1} T_{j_n} < \infty$ may occur ie. infinitely many steps in finite time. We say the process has run out of instructions.

Let $g^n_{ij}(t)$ = pr_i(exactly n jumps to be in state j at time t)

$g^0{}_{ij}(t) = \delta_{ij} e^{-q_i t}$

$g^{m+1}{}_{ij}(t) = \int_0^t e^{-q_i u} \cancel{q_i}\, du \sum_{\alpha \neq i} \frac{q_{i\alpha}}{\cancel{q_i}} g^m_{\alpha j}(t-u)$

$G^{m+1} = G^0 Z G^m$ where the convolution is taken to be implied

in fact:

$G^{m+n+1} = G^n Z G^m$

use induction on n. true for $n=0$

$G^{m+(n+1)+1} = G^{(m+n+1)+1} = G^0 Z G^{m+n+1}$

$= G^0 Z (G^m Z G^n)$

matrix multiplication and convolution are associative so this is:

$= (G^0 Z G^n) Z G^m = G^{n+1} Z G^m$

Let $f^n_{ij}(t)$ = pr_i(at most n jumps to be in state j at time t)

$= \sum_{m \leq n} g^m_{ij}(t)$

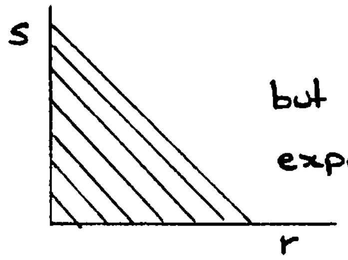
$F^n = G^0 + G^1 + \cdots + G^n$

$G^n(u+v) = \sum_{\substack{r \geq 0 \\ s \geq 0 \\ r+s=n}} G^r(u) G^s(v)$

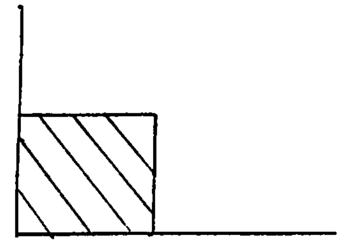
F^n is $\uparrow$ and bounded above, so

$F^n \longrightarrow \Phi$.

$F^n(u+v) = \sum_{m \leq n} \sum_{r+s=m} G^r(u) G^s(v)$



but we might expanded as:



$$\Phi(u+v) = \Phi(u)\Phi(v)$$
$$\Phi(0) = I$$
$$\Phi(u) \geq 0$$

$$G^{n+1} = G^{0} Z G^{n}$$
$$G^{m+1} = G^{m} Z G^{0}$$

$$F^{m+1} = G^{0} + G^{0} Z F^{m}$$
$$= F^{0} + F^{0} Z F^{m}$$
$$F^{n+1} = F^{0} + F^{n} Z F^{0}$$

let $m, n \to \infty$

$$\Phi = F_0 + F^{0} Z \Phi$$
$$= F_0 + \Phi Z F^{0}$$

convolution implied throughout.

$$\Phi_{ij}(t) = \delta_{ij} e^{-q_i t} + \int_0^t e^{-q_i(t-u)} \sum_{\alpha \neq i} q_{i\alpha} \Phi_{\alpha j}(u)\, du$$

$$e^{+q_i t} \Phi_{ij}(t) = \delta_{ij} + \int_0^t \left\{ e^{q_i u} \sum_{\alpha \neq i} q_{i\alpha} \Phi_{\alpha j}(u) \right\} du$$

$$\cancel{e^{q_i t}} \left(q_i \Phi_{ij}(t) + \Phi'_{ij}(t) \right) = \cancel{e^{q_i t}} \sum_{\alpha \neq i} q_{i\alpha} \Phi_{\alpha j}(t)$$

$$\Phi'_{ij}(t) = \sum_{\alpha} q_{i\alpha} \Phi_{\alpha j}(t)$$

$$\frac{d}{dt} \Phi = Q \Phi$$

$$\Phi'_{ij}(0) = Q.$$

$\sum_j f^n_{ij}(t) \leq 1$ let $n \to \infty$

$\sum_j \Phi_{ij}(t) \leq 1$ why < 1 ?

$\Phi_{ij}(t) = \lim_{n\to\infty} pr_i$ (at most n jumps to $X(t) = j$)

$= pr_i$ (finitely many jumps to $X(t) = j$)

$\sum_j \Phi_{ij}(t) = pr_i$ (finitely many jumps to time t)

If < 1 then the process has run out of instruction.

If we do not run out of instructions,

$\sum_j \Phi_{ij}(t) = 1$

Φ is stochastic, and so a Markov process.

$1 - \sum_j \Phi_{ij}(t) = \Delta_i(t)$ - prob. of explosion starting in state i.

Recall condition (c). $\exists$ no bounded solution to $Qy = y$ save $y = 0$

$\Phi = F^0 + F^0 Z \Phi$

$1 - \Delta_i(t) = \sum_j \Phi_{ij}(t) = e^{-q_i t} + \sum \int_0^t e^{-q_i u} du \, q_{i\alpha} \{1 - \Delta_\alpha(t-u)\}$

$\therefore \Delta_i(t) = \sum_{\alpha \neq i} \int_0^t e^{-q_i u} q_{i\alpha} \Delta_\alpha(t-u) \, du$

Def^n: $\Delta_i^*(s) = \int_0^\infty e^{-st} \Delta_i(t) \, du$

$\therefore \Delta_i^*(s) = \sum_{\alpha \neq i} q_{i\alpha} \frac{1}{q_i + s} \Delta_\alpha^*(s)$

$(-q_i + s)\Delta_i^*(s) = \sum_{\alpha \neq i} q_{i\alpha} \Delta_\alpha^*(s) \qquad \forall i$

$s\Delta_i^*(s) = (Q\Delta_i^*(s))_i$

$s = 1$ if $y_i = \Delta_i^*(1)$ then $Qy = y$

y will be bounded.

$$\int_0^\infty e^{-t} \Delta_i(t)\,dt = 0 \qquad \forall i$$

$$\Delta_i(t) = 0 \qquad \sum \Phi_{ij}(t) = 1$$

$\Delta_i^n(t) = 1 - \sum_j f_{ij}^n(t) \longrightarrow \Delta_i(t)$

$F^{n+1} = F^0 + F^0 Z F^n$

$\therefore \quad \Delta_i^{n+1}(t) = \sum_{\alpha \neq i} \int_0^t e^{-q_i u} q_{i\alpha} \Delta_\alpha^n(t-u)\,du$

$\Delta_i^0(t) = 1 - e^{-q_i t} = \int_0^t e^{-q_i u} q_i\,du = \sum_{\alpha \neq i} \int_0^t e^{-q_i u} q_{i\alpha} 1_\alpha\,du$

$\therefore \quad \Delta_i^{-1}(t) = 1 \qquad \forall i, t$ (by convention).

$\Delta_i^{*n+1}(s) = \sum_{\alpha \neq i} q_{i\alpha} \frac{1}{q_i + s} \Delta_i^{*n}(s)$

$\int_0^\infty e^{-st}\, 1\,dt = 1/s \qquad \therefore \quad \Delta_i^{-1}(s) = 1/s.$

$\exists\ y \neq 0$ bounded s.t $Qy = y$

$(q_i + 1) y_i = \sum_{\alpha \neq i} q_{i\alpha} y_\alpha . \qquad y_i = \sum_{\alpha \neq i} q_{i\alpha} \frac{1}{q_i + 1} y_\alpha$

$\leq \sum_{\alpha \neq i} q_{i\alpha} \frac{1}{q_i + 1} 1_\alpha$

$y_i \leq \sum_{\alpha \neq i} q_{i\alpha} \frac{1}{q_i + 1} \Delta^{-1}(1) = \Delta_i^{*0}(1)$

repeating $y_i \leq \Delta_i^{*\circ}(1)$

$0 \le y_i \le \Delta_i^{*n}(1) \quad \forall n \quad n \to \infty$

$1 \ge \Delta^n \downarrow \Delta$

$\therefore \ 0 \le y \le \Delta^*_i(1) = \int_0^\infty e^{-t} \Delta_i(t)\,dt$

For some i and some t, $\Delta_i(t) > 0$.

ie. $\sum_j \Phi_{ij}(t) < 1$

$\therefore$ Starting in j pr(∞ many steps) > 0.

$P(0) = I \qquad P(t) \ge 0$

$P(u+v) = P(u)P(v)$

$P'(t) = QP(t)$

$p_{ij}(t) = e^{-q_i t}\delta_{ij} + \sum_{\alpha \ne i} \int_0^t e^{-q_i u} q_{i\alpha} p_{\alpha j}(t-u)\,du$

$P(t) \ge F_0(t)$

$\quad \ge F_1(t)$

$\therefore \ P(t) \ge F_n(t) \quad \forall t \quad \Rightarrow \quad P(t) \ge \Phi(t)$

$p_{ij}(t) \ge \Phi_{ij}(t)$

$\therefore \ 1 \ge \sum_j p_{ij}(t) \ge \sum_j \Phi_{ij}(t)$

$\therefore$ (c) $\Rightarrow$ $p_{ij}(t) = \Phi_{ij}(t)$ ie. unique solution

Birth and Death process

$q_{i,i+1} = f(i)$

$q_{i,i} = -f(i)$

$q_{ij} = 0$ otherwise

$Qy = y \qquad -f(1)y_1 + f(1)y_2 = y_1$ etc.

$\therefore \ y_{n+1} = \dfrac{1 + f(n)}{f(n)} y_n$.

$$\therefore \quad y_n = y_1 \left(1 + \frac{1}{f(1)}\right)\left(1 + \frac{1}{f(2)}\right) \cdots \left(1 + \frac{1}{f(n-1)}\right)$$

$$= y_n \prod_{1 \le r \le n} \left(1 + \frac{1}{f(r)}\right)$$

(c) is satisfied $\Leftrightarrow$ y_n is divergent

ie. iff $\sum \frac{1}{f(n)} = \infty$

$\therefore$ $\sum \frac{1}{f(n)} < \infty$, population explodes in finite time.

$f(n) = \lambda n$ – o.k. but not if $\lambda n^{1+\epsilon}$

Poisson Process

$$Q = \begin{pmatrix} -\lambda & \lambda & & & 0 \\ & -\lambda & \lambda & & \\ & & -\lambda & \lambda & \\ & 0 & & \ddots & \ddots \end{pmatrix}$$

$\sum \frac{1}{\lambda} = \infty$ so condition (c) is satisfied.

$$p_{0j}(t) = p_{i,j+i}(t)$$
$$p_{00}(t) = e^{-\lambda t}$$
$$p_{0\,n+1}(t) = \int_0^t \lambda p_{0n}(t-u)\, e^{-\lambda u}\, du$$

take $v = t - u$

$$= \int_0^t \lambda p_{0,n}(v)\, e^{-\lambda t} e^{\lambda v}\, dv$$

$$e^{\lambda t} p_{0,n+1}(t) = \int_0^t \lambda p_{0,n}(v)\, e^{\lambda v}\, dv$$

$p_{00}(t) = e^{-\lambda t}$ $\qquad$ $e^{\lambda t} p_{00}(t) = 1$

$p_{01}(t) = \lambda t e^{-\lambda t}$ $\qquad$ $e^{\lambda t} p_{01}(t) = \lambda t$

etc. $\qquad$ $= \frac{1}{2}(\lambda^2 t^2)$

a Poisson $\qquad$ $= \cdots$ etc

$$Q = \begin{pmatrix} \lambda_1 & -\lambda_1 & & & 0 \\ & \lambda_2 & -\lambda_2 & & \\ & & \lambda_3 & -\lambda_3 & \\ & 0 & & \ddots & \ddots \end{pmatrix}$$

$\lambda_n = n\lambda \qquad \sum \frac{1}{\lambda_n} = \frac{1}{\lambda} \sum \frac{1}{n} = \infty \quad : \quad (c).$

let $X(t)$ be the r.v.: state at time t.

$\phi(s,t) = E(s^X)$ be the generating function

$= \sum_{n \geq} s^n \, pr_1(X(t) = n)$

$$\phi(s,t) = e^{-\lambda t} s + \int_0^t \phi(s, t-u)^2 e^{-\lambda u} \lambda \, du$$

put $v = t - u$ and multiply be $e^{\lambda t}$

$$e^{\lambda t} \phi = s + \int_0^t \phi^2 e^{\lambda v} \lambda \, dv$$

$$e^{\lambda t} \left(\lambda \phi + \frac{\partial \phi}{\partial t} \right) = \phi^2 e^{\lambda t} \lambda$$

$$\partial \phi / \partial t = \lambda \phi (\phi - 1)$$

$$\frac{d\phi}{\phi(\phi - 1)} = \lambda dt$$

$$\left(\frac{1}{\phi - 1} - \frac{1}{\phi} \right) d\phi = \lambda dt$$

$\log \left(\frac{\phi - 1}{\phi} \right) = \lambda t + c(s)$ and $\phi(s, 0) = s$

so $\log \left(\frac{\phi - 1}{\phi} \right) = \lambda t + \log \left(\frac{s-1}{s} \right)$

$$\phi(s,t) = \frac{s e^{-\lambda t}}{1 - s(1 - e^{-\lambda t})}$$

$$= s e^{-\lambda t} \sum_{n \geq 0} s^n (1 - e^{-\lambda t})^n$$

$p_{1n}(t)$ is the coeff of $s^n = e^{-\lambda t} (1 - e^{-\lambda t})^{n-1}$

$p_{m,n}(t)$: $X(0) = m$?

ie. a population starting with m individuals.

$\phi_m(s,t) = \{\phi(s,t)\}^m$.

$= s^m e^{-m\lambda t} / [1 - s(1-e^{-\lambda t})]^m$ gives

$$P_{m,n}(t) = e^{-\lambda m t} \frac{m(m+1)\cdots(n-1)}{(n-m)!} (1-e^{-\lambda t})^{n-m}$$

Death process

$$Q = \begin{pmatrix} 0 & 0 & 0 & \cdots & & \\ \mu & -(\lambda+\mu) & \lambda & & & \\ & 2\mu & -2(\lambda+\mu) & 2\lambda & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots \end{pmatrix}$$

$Qy = y$: $0 \le y \le 1$

$y_0 = 0$

$0 - (\lambda+\mu)y_1 + \lambda y_2 = y_1$ $\quad y_1 = 0 \Rightarrow y_2 = y_3 = \cdots = 0$

so we must have $y_1 \ne 0$ say $y_1 = 1$

$$\mu y_{j-1} - (\lambda+\mu) y_j + \lambda y_{j+1} = \frac{y_j}{j}$$

$$\lambda(y_{j+1} - y_j) = \mu(y_j - y_{j-1}) + \frac{y_j}{j}$$

$y_0 = 0$ $\quad y_1 = 1$ $\quad y_2 = \dfrac{1+\lambda+\mu}{\lambda} > 1$.

$y_0 < y_1 < y_2 \cdots < y_n$ by induction.

$y_n \uparrow z \le \infty$.

$$(\lambda+\mu) y_{j+1} = \lambda y_j + \mu y_j - \mu y_{j-1} + \frac{y_j}{j} + \mu y_{j+1}$$

$$> (\lambda+\mu) y_j + \frac{y_j}{j}$$

$$y_{j+1} > \left(1 + \frac{1}{(\lambda+\mu)j}\right) y_j$$

$$y_{j+1} > \prod_{n=1}^{j} \left(1 + \frac{1}{n(\lambda+\mu)\hat{n}} \right)$$

$$z \geqslant \prod_{1}^{\infty} \left(1 + \frac{1}{n(\lambda+\mu)} \right) = \infty$$

so the birth-death process is regular.

$0 \leq s \leq 1 \qquad E s^X = \phi(s,t).$

$X(0) = 1$

$$\phi(s,t) = e^{-(\lambda+\mu)t} s + \int_0^t e^{-(\lambda+\mu)u} (\lambda+\mu)\, du \left[\frac{\lambda}{\lambda+\mu} \{\phi(s,t-u)\}^2 + \frac{\mu}{\lambda+\mu} \right]$$

$v = t-u$, multiplying be $e^{(\lambda+\mu)t}$

diff. wrt t to get:

$$\left((\lambda+\mu)\phi + \partial\phi/\partial t \right) e^{(\lambda+\mu)t} = (\lambda\phi^2 + \mu) e^{(\lambda+\mu)t}$$

$$d\phi/dt = \lambda\phi^2 - (\lambda+\mu)\phi + \mu$$

$$\frac{d\phi}{(\lambda\phi-\mu)(\phi-1)} = dt$$

$$\left(\frac{1}{\phi-1} - \frac{\lambda}{\lambda\phi-\mu} \right) d\phi = (\lambda-\mu)\, dt$$

$\lambda \neq \mu$ then

$$\log\left(\frac{\phi-1}{\lambda\phi-\mu} \right) = (\lambda-\mu)t + \log\left(\frac{s-1}{\lambda s-\mu} \right)$$

$$\frac{\phi-1}{\lambda\phi-\mu} = \frac{(s-1)}{\lambda s-\mu} E \qquad E = e^{(\lambda-\mu)t}$$

$$\phi(s,t) = \frac{(\lambda s-\mu) - \mu(s-1)e^{(\lambda-\mu)t}}{(\lambda s-\mu) - \lambda(s-1)e^{(\lambda-\mu)t}}$$

$$\phi(s,t) = \frac{\mu E - \lambda}{\lambda(E-1)} + \frac{(\lambda-\mu)^2 E}{(\lambda E-\mu)\lambda(E-1)}\left\{1 - \frac{\lambda(E-1)}{\lambda E-\mu} s\right\}^{-1}$$

$$\phi(s,t) = P + (1-P)\frac{(1-Q)}{(1-Qs)}$$

$$P_{1,0}(t) = P + (1-P)(1-Q)$$
$$P_{1,n}(t) = (1-P)(1-Q)Q^{n-1} \qquad n \geq 1.$$

$$\phi_i(s,t) = (E s^X)^i$$
$$= \left\{\frac{\mu(E-1) - s(\mu E-\lambda)}{(\lambda E-\mu) - \lambda s(E-1)}\right\}^i$$

$$pr_i(X(t)=0) = \left\{\frac{\mu(e^{(\lambda-\mu)t} - 1)}{\lambda e^{(\lambda-\mu)t} - \mu}\right\}^i$$

$\lambda > \mu \quad \rightarrow (\mu/\lambda)^i$ as $t \rightarrow \infty$

$\lambda < \mu \quad \rightarrow 1$ as $t \rightarrow \infty$

when

$\lambda = \mu \quad \rightarrow 1$

note in the supercritical case there is a positive possibility of extinction.

$$E(X(t)) = X(0)e^{(\lambda-\mu)t}$$

When $\lambda > \mu$ the time to distinction T is as above.

$$X(t)/(X(0)e^{(\lambda-\mu)t}) \rightarrow W \quad \text{as } t \rightarrow \infty$$
$$X(t) = X(0)We^{(\lambda-\mu)t} + o(e^{(\lambda-\mu)t})$$

$$\{\underset{\uparrow \text{r.v.}}{X(t)} \mid {}^{X(t)>0}_{T>t}\} = \text{const.} + \ln W + (\lambda-\mu)t$$
$$= (\lambda-\mu)(t - \underset{\uparrow \text{r.v}}{\Delta})$$

Epidemics

x : susceptibles
y : infectious
z : removables (had disease, died, quarenteened)
N

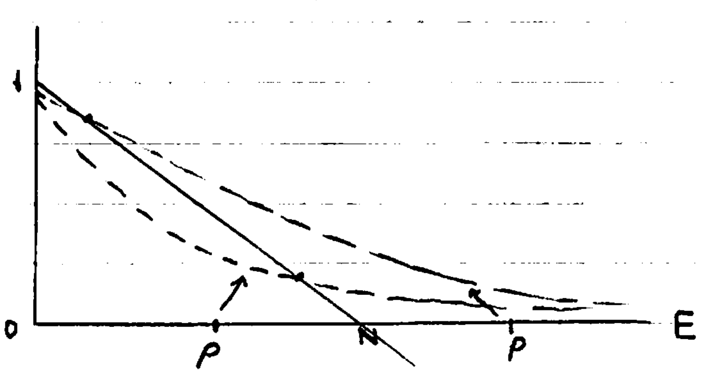
$$\frac{dx}{dt} = -\beta x y$$

$$\frac{dy}{dt} = \beta x y - \gamma y$$

$\dot{x} = -xy$
$\dot{y} = xy - \rho y$ $\quad \rho = \gamma/\beta$ changing time scale.
intially N suseptibles, 1 infectious: $\psi =$
$x + y + \rho \ln\left(\frac{N}{x}\right)$ has derivative zero.

	before	after
x	N	$N-E$
y	1	0
z	0	$E+1$
$N+1$	$N+1$	

before
$$\psi = N + 1 + \rho \ln\left(\frac{N}{N}\right) = N + 1$$
after
$$\psi = N - E + \rho \ln\left(\frac{N}{N-E}\right)$$
so
$$\frac{E}{N} - 1 = e^{-3} e^{-E/\rho}$$



$$\lim_{\delta \to 0} E = \begin{cases} 0 & N \leq \rho \\ \text{+ve root of } 1 - \frac{E}{N} = e^{-E/\rho} & N > \rho \end{cases}$$